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Blow-up behaviors for semilinear parabolic systems coupled in equations and boundary conditions ☆

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Abstract

This paper concerns with blow-up behaviors for semilinear parabolic systems coupled in equations and boundary conditions in half space. We establish the rate estimates for blow-up solutions and prove that the blow-up set is $\partial\mathbb{R}_+^N$ under proper conditions on initial data. Furthermore, for $N = 1$, more complete conclusions about such two topics are given.

Keywords: Semilinear parabolic systems; Nonlinear boundary conditions; Blow-up rate estimates; Upper and lower bounds; Blow-up sets

1. Introduction and main results

In this paper, we study the estimates of blow-up rate and blow-up set of positive solutions to the following semilinear parabolic systems with nonlinear boundary conditions:

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$$\begin{cases} u_t = \Delta u + v^p, \quad v_t = \Delta v + u^k, & x \in \mathbb{R}_+^N,\ t > 0, \\ -\frac{\partial u}{\partial x_1} = v^q, \quad -\frac{\partial v}{\partial x_1} = u^m, & x_1 = 0,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}_+^N, \end{cases} \tag{1.1}$$

where $\mathbb{R}_+^N = \{(x_1, x') \mid x_1 > 0,\ x' \in \mathbb{R}^{N-1}\}$, p, q, k and m are positive constants. Initial data $u_0(x)$ and $v_0(x)$ are bounded nonnegative and nontrivial C^1 functions and satisfy

$$\begin{cases} -\frac{\partial u_0}{\partial x_1} = v_0^q, \quad -\frac{\partial v_0}{\partial x_1} = u_0^m \quad \text{for } x_1 = 0, \\ \lim_{|x|\to\infty} u_0(x) = \lim_{|x|\to\infty} v_0(x) = 0, \\ \frac{\partial u_0}{\partial x_1} \leqslant 0, \quad \frac{\partial v_0}{\partial x_1} \leqslant 0, \quad x \in \mathbb{R}_+^N. \end{cases} \tag{1.2}$$

It follows from the classical results that the solution $(u,\ v)$ of (1.1) satisfies

$$u_{x_1}, v_{x_1} \leqslant 0, \quad \lim_{|x|\to\infty} u(x,t) = \lim_{|x|\to\infty} v(x,t) = 0, \quad x \in \mathbb{R}_+^N,\ t \in (0,T), \tag{1.3}$$

T being the time of existence.

By the maximum principle and the results of [5–8], we know that if

$$\max\{pk, mp, mq, qk\} > 1, \tag{1.4}$$

then the solution (u, v) of (1.1) blows up in finite time for suitable "large" initial data. Throughout this paper we assume that (1.4) holds and the solution (u, v) of (1.1) blows up in finite time T. It is obvious that u and v blow up simultaneously.

Our work on blow-up rate estimates is motivated by papers [2,5,27]. In paper [2], Chlebik and Fila studied upper bounds of blow-up rates of solutions to the following problems:

$$\begin{cases} u_t = \Delta u + v^p, \quad v_t = \Delta v + u^k, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, & x \in \mathbb{R}^N, \\ u_0, v_0 \in L^\infty(\mathbb{R}^N), \end{cases} \tag{1.5}$$

and

$$\begin{cases} u_t = \Delta u + \delta_1 v^p, \quad v_t = \Delta v + \delta_2 u^k, & x \in \mathbb{R}_+^N,\ t > 0, \\ -u_{x_1} = \delta_3 v^q, \quad -v_{x_1} = \delta_4 u^m, & x_1 = 0,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, & x \in \mathbb{R}_+^N, \\ u_0, v_0 \in L^\infty(\mathbb{R}_+^N), \end{cases} \tag{1.6}$$

where constants $\delta_i \in \{0, 1\}$. Let T be the blow-up time of solution (u, v) and C be a suitable positive constant. Their results read as follows.

For problem (1.5), if $pk > 1$ and $\max\{1+p, 1+k\}/(pk-1) \geqslant N/2$, then

$$u(x,t) \leqslant C(T-t)^{-(p+1)/(pk-1)},$$
$$v(x,t) \leqslant C(T-t)^{-(k+1)/(pk-1)}, \quad 0 < t < T.$$

For problem (1.6). When $\delta_1 = \delta_2 = 0$ and $\delta_3 = \delta_4 = 1$, if $mq > 1$ and $\max\{1+m, 1+q\}/(mq-1) \geqslant N$, then

$$u(x,t) \leqslant C(T-t)^{-(1+q)/[2(mq-1)]},$$
$$v(x,t) \leqslant C(T-t)^{-(1+m)/[2(mq-1)]}, \quad 0 < t < T.$$

When $\delta_1 = \delta_4 = 1$ and $\delta_2 = \delta_3 = 0$, if $mp > 1$ and $u_{x_1}, v_{x_1} \leqslant 0$, and if moreover, one of the following holds:

(i) $\max\{p+2, 2m+1\}/(mp-1) > N$,
(ii) $\max\{p+2, 2m+1\}/(mp-1) = N$ and $m, p \geqslant 1$,

then

$$u(x,t) \leqslant C(T-t)^{-(p+2)/[2(mp-1)]},$$
$$v(x,t) \leqslant C(T-t)^{-(2m+1)/[2(mp-1)]}, \quad 0 < t < T.$$

For the case $\delta_1 = \delta_2 = 0$, $\delta_3 = \delta_4 = 1$ and $N = 1$, the authors of [27] obtained lower bounds of blow-up rates of solutions to problem (1.6), and improved the results of paper [5] by removing the restriction $\min\{m, q\} \geqslant 1$ and allowing a larger class of solutions.

Our work on the blow-up set estimate is inspired by papers [9–14,16]. In papers [9–14], the authors studied the equation

$$u_t = \Delta u + u^p, \quad p > 1,$$

and gave the blow-up set estimate (including the blow-up rate estimate and the profile of solution as t tends to the blow-up time). In [14] (the domain $\Omega = B_R(0)$) and [16], the authors discussed the following single heat equation with nonlinear boundary condition:

$$\begin{cases} u_t = \Delta u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \eta} = u^p, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geqslant 0, & x \in \Omega \end{cases}$$

with $p > 1$, and proved that the blow-up occurs only on the boundary of the domain.

An important related work is the paper [24]. In [24], the second author of the present paper discussed the following semilinear parabolic systems with nonlinear boundary conditions:

$$\begin{cases} u_t = u_{xx} + v^p, \quad v_t = v_{xx} + u^k, & 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = v^q(1,t), & t > 0, \\ v_x(0,t) = 0, \quad v_x(1,t) = u^m(1,t), & t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & 0 \leqslant x \leqslant 1. \end{cases} \tag{1.7}$$

He established the explicit description of the effects of reaction terms and nonlinear boundary conditions on the blow-up rates of solutions to system (1.7), and proved that the blow-up occurs only at the boundary $x = 1$.

Compared with our problem (1.1), problems (1.5) and (1.6) have only two nonlinear terms, and the initial-boundary value problem of (1.7) is concerned in a bounded interval $[0, 1]$ although there are four nonlinear terms in this problem. Therefore, it seems interesting and important to extend the results of [2,24,27] to our present problem (1.1) and give more perfect conclusions. Roughly speaking, our problem (1.1) is coupled with two nonlinear reaction terms and two nonlinear boundary conditions and is discussed in the half space (higher dimension). Furthermore, besides upper bounds of blow-up rate and the description of blow-up set just as in [2], we also provide the lower bounds of blow-up rate, and in addition, for $N = 1$, more complete conclusions about the blow-up rate and the blow-up set are presented.

The main purposes of this paper are to give an explicit description of the effects of reaction terms and nonlinear boundary conditions on the blow-up rates of solutions to system (1.1), and then to verify that blow-up occurs only on the boundary $\partial\mathbb{R}^N_+$. Our main results are as follows.

Theorem 1. *There exists a constant $c > 0$ such that the following hold.*

(i) *When $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k \geqslant (2mp + 2m - 1)/(2 + p)$, we have*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} u(x,\tau) \geqslant c(T-t)^{-(1+p)/(pk-1)}, & 0<t<T, \\ \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} v(x,\tau) \geqslant c(T-t)^{-(1+k)/(pk-1)}, & 0<t<T. \end{cases} \tag{1.8}$$

(ii) *When $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k < (2mp + 2m - 1)/(2 + p)$. If $p \geqslant (2mq + q - 1)/(1 + m)$, then*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} u(x,\tau) \geqslant c(T-t)^{-(2+p)/[2(mp-1)]}, & 0<t<T, \\ \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} v(x,\tau) \geqslant c(T-t)^{-(1+2m)/[2(mp-1)]}, & 0<t<T; \end{cases} \tag{1.9}$$

if $p < (2mq + q - 1)/(1 + m)$, then

$$\begin{cases} \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} u(x,\tau) \geqslant c(T-t)^{-(1+q)/[2(mq-1)]}, & 0<t<T, \\ \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} v(x,\tau) \geqslant c(T-t)^{-(1+m)/[2(mq-1)]}, & 0<t<T. \end{cases} \tag{1.10}$$

(iii) *When $p < (2qk + 2q - 1)/(2 + k)$ and $k \geqslant (2mp + 2m - 1)/(2 + p)$. If $k \geqslant (2mq + m - 1)/(1 + q)$, then*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} u(x,\tau) \geqslant c(T-t)^{-(1+2q)/[2(qk-1)]}, & 0<t<T, \\ \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} v(x,\tau) \geqslant c(T-t)^{-(2+k)/[2(qk-1)]}, & 0<t<T; \end{cases} \tag{1.11}$$

if $k < (2mq + m - 1)/(1 + q)$, then (1.10) *holds.*

(iv) *When $p < (2qk + 2q - 1)/(2 + k)$ and $k < (2mp + 2m - 1)/(2 + p)$,* (1.10) *holds.*

Theorem 2. *There exists a constant $C > 0$ such that the following hold.*

(i) *Assume that $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k \geqslant (2mp + 2m - 1)/(2 + p)$. If $\max\{1 + p, 1 + k\}/(pk - 1) \geqslant N/2$, then*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} u(x,\tau) \leqslant C(T-t)^{-(1+p)/(pk-1)}, & 0<t<T, \\ \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} v(x,\tau) \leqslant C(T-t)^{-(1+k)/(pk-1)}, & 0<t<T. \end{cases} \tag{1.12}$$

(ii) *Assume that $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k < (2mp + 2m - 1)/(2 + p)$. For the case $p \geqslant (2mq + q - 1)/(1 + m)$, if $\max\{p + 2, 1 + 2m\}/(mp - 1) > N$, or $\max\{p + 2, 1 + 2m\}/(mp - 1) = N$ and $m, p \geqslant 1$, then*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} u(x,\tau) \leqslant C(T-t)^{-(2+p)/[2(mp-1)]}, & 0<t<T, \\ \max_{0\leqslant\tau\leqslant t}\sup_{\mathbb{R}^N_+} v(x,\tau) \leqslant C(T-t)^{-(1+2m)/[2(mp-1)]}, & 0<t<T; \end{cases} \tag{1.13}$$

for the case $p < (2mq + q - 1)/(1 + m)$, *if* $\max\{1 + q, 1 + m\}/(mq - 1) \geqslant N$, *then*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t} \sup_{\mathbb{R}_+^N} u(x,\tau) \leqslant C(T-t)^{-(1+q)/[2(mq-1)]}, & 0 < t < T, \\ \max_{0\leqslant\tau\leqslant t} \sup_{\mathbb{R}_+^N} v(x,\tau) \leqslant C(T-t)^{-(1+m)/[2(mq-1)]}, & 0 < t < T. \end{cases} \tag{1.14}$$

(iii) *Assume that* $p < (2qk + 2q - 1)/(2 + k)$ *and* $k \geqslant (2mp + 2m - 1)/(2 + p)$. *For the case* $k \geqslant (2mq + m - 1)/(1 + q)$, *if* $\max\{1 + 2q, 2 + k\}/(qk - 1) > N$, *or* $\max\{1 + 2q, 2 + k\}/(qk - 1) = N$ *and* $q, k \geqslant 1$, *then*

$$\begin{cases} \max_{0\leqslant\tau\leqslant t} \sup_{\mathbb{R}_+^N} u(x,\tau) \leqslant C(T-t)^{-(1+2q)/[2(qk-1)]}, & 0 < t < T, \\ \max_{0\leqslant\tau\leqslant t} \sup_{\mathbb{R}_+^N} v(x,\tau) \leqslant C(T-t)^{-(2+k)/[2(qk-1)]}, & 0 < t < T; \end{cases} \tag{1.15}$$

for the case $k < (2mq + m - 1)/(1 + q)$, *if* $\max\{1 + q, 1 + m\}/(mq - 1) \geqslant N$, *then* (1.14) *holds.*

(iv) *Assume that* $p < (2qk + 2q - 1)/(2 + k)$ *and* $k < (2mp + 2m - 1)/(2 + p)$. *If* $\max\{1 + q, 1 + m\}/(mq - 1) \geqslant N$, *then* (1.14) *holds.*

Theorem 3. *For the cases* (ii), (iii) *and* (iv) *of Theorem* 2, *blow-up occurs only on the boundary* $\partial\mathbb{R}_+^N$. *More precisely, if* $\Omega_0 \subset \mathbb{R}_+^N$ *is such that* $\bar{\Omega}_0 \subset \mathbb{R}_+^N$, *then*

$$\sup_{0\leqslant t<T} \left\{ \left\| u(\cdot,t) \right\|_{C(\bar{\Omega}_0)} + \left\| v(\cdot,t) \right\|_{C(\bar{\Omega}_0)} \right\} < \infty.$$

Remark 1. In Theorems 2 and 3, the assumptions depend on the dimension N, so the conclusions are not complete. However, we do not know how to relax these assumptions. Fortunately, in the case $N = 1$, we can do this, see Section 5.

Remark 2. If the case (i) of Theorem 2 occurs, we do not know whether or not the conclusion of Theorem 3 is true. It is still an open problem.

There are many related works on the blow-up rates of solutions to parabolic systems with nonlinear boundary conditions, please refer to [1,3,4,10,15,17,19–23,25,26] and references therein.

The organization of this paper is as follows. In Sections 2–4, we give the proofs of Theorems 1–3, respectively. In Section 5, we discuss further the upper bounds of blow-up rates and the blow-up set for the case $N = 1$, and give more complete conclusions on the upper bounds of blow-up rates and on the blow-up set.

2. Proof of Theorem 1

In this section, we will give lower bounds of blow-up rates of solutions to system (1.1), that is, to prove Theorem 1. We begin with three lemmas. For convenience, denote

$$f(t) = \max_{0\leqslant\tau\leqslant t} \sup_{\mathbb{R}_+^N} u(x,\tau), \quad g(t) = \max_{0\leqslant\tau\leqslant t} \sup_{\mathbb{R}_+^N} v(x,\tau), \quad t \in [0,T), \tag{2.1}$$

then $f(t)$ and $g(t)$ are nondecreasing in t. We first apply the ideas of [16] to prove the following lemma.

Lemma 1. *Let α, β be positive constants and satisfy*

$$\begin{cases} 2+\alpha-p\beta \geqslant 0, & 2+\beta-k\alpha \geqslant 0, \\ 1+\alpha-q\beta \geqslant 0, & 1+\beta-m\alpha \geqslant 0. \end{cases} \tag{2.2}$$

If one of the following holds:

(a) $2+\alpha-p\beta = 2+\beta-k\alpha = 0$,
(b) $2+\alpha-p\beta = 1+\beta-m\alpha = 0$,
(c) $1+\alpha-q\beta = 2+\beta-k\alpha = 0$,
(d) $1+\alpha-q\beta = 1+\beta-m\alpha = 0$,

then there exists a positive constant ε such that

$$\varepsilon g^{1/\beta}(t) \leqslant f^{1/\alpha}(t), \quad \varepsilon f^{1/\alpha}(t) \leqslant g^{1/\beta}(t), \quad \forall t \in [T/2, T). \tag{2.3}$$

Proof. On the contrary we assume that the first inequality of (2.3) is not true, then there exists a sequence $\{t_n\}$ with $t_n \to T^-$ as $n \to \infty$ such that

$$g^{-1/\beta}(t_n) f^{1/\alpha}(t_n) \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

For each t_n, in view of (1.3) and the definition of g, we can choose $(\hat{x}_n, \hat{t}_n) \in \partial\mathbb{R}_+^N \times (0, t_n]$ such that $v(\hat{x}_n, \hat{t}_n) \geqslant g(t_n)/2$. Since $g(t_n) \to \infty$, it follows that $\hat{t}_n \to T^-$. Let

$$\begin{cases} \lambda_n = \lambda(t_n) = g^{-1/\beta}(t_n), \\ \varphi_n(y,s) = \lambda_n^\alpha u(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n), & y \in \mathbb{R}_+^N,\ s \in I_n(T), \\ \psi_n(y,s) = \lambda_n^\beta v(\lambda_n y + \hat{x}_n, \lambda_n^2 s + \hat{t}_n), & y \in \mathbb{R}_+^N,\ s \in I_n(T), \end{cases} \tag{2.5}$$

where $I_n(t) = (-\lambda_n^{-2}\hat{t}_n, \lambda_n^{-2}(t - \hat{t}_n))$. Direct computations show that φ_n and ψ_n satisfy

$$\begin{cases} (\varphi_n)_s = \Delta_y \varphi_n + \lambda_n^{2+\alpha-p\beta} \psi_n^p, & y \in \mathbb{R}_+^N,\ s \in I_n(T), \\ (\psi_n)_s = \Delta_y \psi_n + \lambda_n^{2+\beta-k\alpha} \varphi_n^k, & y \in \mathbb{R}_+^N,\ s \in I_n(T), \\ -(\varphi_n)_{y_1} = \lambda_n^{1+\alpha-q\beta} \psi_n^q, & y_1 = 0,\ s \in I_n(T), \\ -(\psi_n)_{y_1} = \lambda_n^{1+\beta-m\alpha} \varphi_n^m, & y_1 = 0,\ s \in I_n(T), \end{cases} \tag{2.6}$$

and

$$\begin{cases} \psi_n(0,0) \geqslant 1/2, \quad 0 \leqslant \psi_n(y,s) \leqslant 1, & y \in \overline{\mathbb{R}_+^N},\ s \in (-\lambda_n^{-2}\hat{t}_n, 0], \\ 0 \leqslant \varphi_n(y,s) \leqslant f(t_n) g^{-\alpha/\beta}(t_n), & y \in \overline{\mathbb{R}_+^N},\ s \in (-\lambda_n^{-2}\hat{t}_n, 0]. \end{cases} \tag{2.7}$$

In view of (2.2), (2.4), (2.7) and $\lambda_n \to 0$, we know that the nonlinear terms in (2.6) are all uniformly bounded. For any $K > 0$, by (2.6) and the Schauder's estimates (cf. [18]) we deduce that

$$\|(\varphi_n, \psi_n)\|_{C^{2+\mu,1+\mu/2}(\{\overline{\mathbb{R}_+^N} \cap \{|y| \leqslant K\}\} \times [-K,0])} \leqslant C_K,$$

where the constant C_K is independent of n. It follows that there exists a subsequence of $\{(\varphi_n, \psi_n)\}$, which is also denoted by $\{(\varphi_n, \psi_n)\}$, and nonnegative functions φ and ψ such that

$$(\varphi_n, \psi_n) \to (\varphi, \psi) \quad \text{locally uniformly on } [0, K] \times [-K, 0],$$

and (φ, ψ) satisfies

$$\begin{cases} \varphi_s = \Delta_y \varphi + \delta_1 \psi^p, \quad \psi_s = \Delta_y \psi + \delta_2 \varphi^k, & y \in \mathbb{R}^N_+,\ s \in (-\infty, 0], \\ -\varphi_{y_1} = \delta_3 \psi^q, \quad -\psi_{y_1} = \delta_4 \varphi^m, & y_1 = 0,\ s \in (-\infty, 0], \end{cases} \tag{2.8}$$

where δ_i, $i = 1, 2, 3, 4$, are nonnegative constants and satisfy

$$\begin{cases} \delta_1 = \delta_2 = 1, & \text{if } 2 + \alpha - p\beta = 2 + \beta - k\alpha = 0, \\ \delta_1 = \delta_4 = 1, & \text{if } 2 + \alpha - p\beta = 1 + \beta - m\alpha = 0, \\ \delta_2 = \delta_3 = 1, & \text{if } 1 + \alpha - q\beta = 2 + \beta - k\alpha = 0, \\ \delta_3 = \delta_4 = 1, & \text{if } 1 + \alpha - q\beta = 1 + \beta - m\alpha = 0. \end{cases} \tag{2.9}$$

We should point out that some δ_i may be zero. For example, if $2 + \alpha - p\beta > 0$, then $\delta_1 = 0$ since $\lambda_n \to 0$. It is clear that φ and ψ are continuous at $(0, y'; 0)$ for $y' \in \mathbb{R}^{N-1}$. Applying (2.4) and (2.7), we find that $\varphi(y, s) \equiv 0$, $\psi(0, 0) \geqslant 1/2$. This is a contradiction to (2.8) and (2.9).

In a similar way, we can prove the second inequality of (2.3). $\square$

The lemma below plays a key role in the proof of the lower bounds.

Lemma 2. *Under the assumptions of Lemma* 1*, we have*

(A) *If $p \geqslant 2q - \alpha/\beta$, then*

$$f(z) \geqslant c(T - z)^{-1/(p\beta/\alpha - 1)},$$
$$g(z) \geqslant c(T - z)^{-1/(p - \alpha/\beta)}, \quad T/2 < z < T. \tag{2.10}$$

(B) *If $p < 2q - \alpha/\beta$, then*

$$f(z) \geqslant c(T - z)^{-1/[2(q\beta/\alpha - 1)]},$$
$$g(z) \geqslant c(T - z)^{-1/[2(q - \alpha/\beta)]}, \quad T/2 < z < T. \tag{2.11}$$

Proof. Recall that the Green's function $G(x; y; t)$ for the heat equation in $\mathbb{R}^N_+$ with $\partial G/\partial y_1 = 0$ at $y_1 = 0$ is given by

$$G(x; y; t) = (4\pi t)^{-N/2} \exp\left(-\frac{|x' - y'|^2}{4t}\right) \times \left\{\exp\left(-\frac{(x_1 - y_1)^2}{4t}\right) + \exp\left(-\frac{(x_1 + y_1)^2}{4t}\right)\right\}.$$

For any $0 \leqslant z < t < T$, we have Green's identity (see [5])

$$u(x, t) = \int_{\mathbb{R}^N_+} G(x; y; t - z) u(y; z)\, dy + \int_z^t \int_{\mathbb{R}^N_+} G(x; y; t - \tau) v^p(y; \tau)\, dy\, d\tau + \int_z^t \int_{\mathbb{R}^{N-1}} G(x; 0, y'; t - \tau) v^q(0, y'; \tau)\, dy'\, d\tau. \tag{2.12}$$

Note that $\int_{\mathbb{R}_+^N} G(x; y; t-z)\,dy = 1$ for $0 < z < t$, and $g(t)$ is nondecreasing in t. In view of (2.12) and the first inequality of (2.3),

$$\begin{aligned} f(t) &\leqslant f(z) + (t-z)g^p(t) + \pi^{-1/2}\int_z^t (t-\tau)^{-1/2} g^q(t)\,d\tau \\ &\leqslant f(z) + (T-z)g^p(t) + 2\pi^{-1/2}\sqrt{T-z}\,g^q(t) \\ &\leqslant f(z) + C\bigl\{(T-z)f^{p\beta/\alpha}(t) + \sqrt{T-z}\,f^{q\beta/\alpha}(t)\bigr\}, \end{aligned} \tag{2.13}$$

where C is a positive constant. Since $f(t) \to \infty$ as $t \to T^-$, for any $z \in (T/2, T)$, one can choose t: $0 < z < t < T$ such that $f(t) = 2f(z)$. Without loss of generality we may assume that $f(t), g(t) > 1$ for $T/2 < z < T$.

(A) If $p \geqslant 2q - \alpha/\beta$, i.e., $p\beta/\alpha \geqslant 2q\beta/\alpha - 1$. Then we have, by (2.13),

$$\begin{aligned} f(z) &\leqslant C\bigl\{(T-z)f^{p\beta/\alpha}(z) + (T-z)^{1/2} f^{(1+p\beta/\alpha)/2}(z)\bigr\} \\ &\leqslant C\bigl\{(T-z)f^{p\beta/\alpha}(z) + C(\varepsilon)(T-z)f^{p\beta/\alpha}(z) + \varepsilon f(z)\bigr\} \\ &\leqslant C(T-z)f^{p\beta/\alpha}(z), \quad T/2 < z < T. \end{aligned}$$

This implies the first inequality of (2.10). Using (2.3), we obtain the second one of (2.10).

(B) If $p < 2q - \alpha/\beta$, i.e., $p\beta/\alpha < 2q\beta/\alpha - 1$. Then we have, by (2.13),

$$\begin{aligned} f(z) &\leqslant C\bigl\{(T-z)f^{2q\beta/\alpha-1}(z) + (T-z)^{1/2} f^{q\beta/\alpha}(z)\bigr\} \\ &\leqslant C\bigl\{(T-z)f^{2q\beta/\alpha-1}(z) + (T-z)^{1/2} f^{q\beta/\alpha-1/2}(z) f^{1/2}(z)\bigr\} \\ &\leqslant C\bigl\{(T-z)f^{2q\beta/\alpha-1}(z) + C(\varepsilon)(T-z)f^{2q\beta/\alpha-1}(z) + \varepsilon f(z)\bigr\} \\ &\leqslant C(T-z)f^{2q\beta/\alpha-1}(z), \quad T/2 < z < T. \end{aligned}$$

The first inequality of (2.11) holds. Consequently, the second inequality of (2.11) holds by (2.3). The proof of Lemma 2 is completed. □

The following lemma was given by [24].

Lemma 3 [24, Lemma 2].

(1°) *If* $p \geqslant (2qk + 2q - 1)/(2 + k)$ *and* $k \geqslant (2mp + 2m - 1)/(2 + p)$, *then* $pk > 1$.
(2°) *If* $p \geqslant \max\{(2qk+2q-1)/(2+k), (2mq+q-1)/(1+m)\}$ *and* $k < (2mp+2m-1)/(2+p)$, *then* $mp > 1$.
(3°) *If* $(2qk + 2q - 1)/(2 + k) \leqslant p < (2mq + q - 1)/(1 + m)$ *and* $k < (2mp + 2m - 1)/(2 + p)$, *then* $k < (2mq + m - 1)/(1 + q)$ *and* $mq > 1$.
(4°) *If* $k \geqslant \max\{(2mp + 2m - 1)/(2 + p), (2mq + m - 1)/(1 + q)\}$ *and* $p < (2qk + 2q - 1)/(2 + k)$, *then* $qk > 1$.
(5°) *If* $(2mp + 2m - 1)/(2 + p) \leqslant k < (2mq + m - 1)/(1 + q)$ *and* $p < (2qk + 2q - 1)/(2 + k)$, *then* $p < (2mq + q - 1)/(1 + m)$ *and* $mq > 1$.
(6°) *If* $p < (2qk + 2q - 1)/(2 + k)$ *and* $k < (2mp + 2m - 1)/(2 + p)$, *then*

$$p < (2mq + q - 1)/(1 + m), \qquad k < (2mq + m - 1)/(1 + q)$$

and $mq > 1$.

Now, we are able to prove Theorem 1.

Proof of Theorem 1. (i) If $p \geqslant (2qk+2q-1)/(2+k)$ and $k \geqslant (2mp+2m-1)/(2+p)$, then $pk > 1$ by Lemma 3(1°). Let

$$\alpha = 2(1+p)/(pk-1), \qquad \beta = 2(1+k)/(pk-1).$$

Series of computations imply that (2.2) and Lemma 1(a) hold. Moreover, $p \geqslant 2q - \alpha/\beta$ since $p \geqslant (2qk+2q-1)/(2+k)$. Therefore, (2.10) holds by Lemma 2. A direct calculation gives

$$p\beta/\alpha - 1 = (pk-1)/(1+p), \qquad p - \alpha/\beta = (pk-1)/(1+k).$$

Hence, (2.1) and (2.10) yield (1.8).

(ii) When $p \geqslant (2qk+2q-1)/(2+k)$ and $k < (2mp+2m-1)/(2+p)$. If $p \geqslant (2mq+q-1)/(1+m)$, then $mp > 1$ by Lemma 3(2°). Denote

$$\alpha = (2+p)/(mp-1), \qquad \beta = (1+2m)/(mp-1).$$

A direct calculation tells us that (2.2) and Lemma 1(b) hold. The condition $p \geqslant (2mq+q-1)/(1+m)$ implies $p \geqslant 2q - \alpha/\beta$. By Lemma 2, (2.10) holds. A simple computation shows that

$$p\beta/\alpha - 1 = 2(mp-1)/(2+p), \qquad p - \alpha/\beta = 2(mp-1)/(1+2m).$$

Combing (2.10) and (2.1) gets (1.9).

If $p < (2mq+q-1)/(1+m)$, then $k < (2mq+m-1)/(1+q)$ and $mq > 1$ by Lemma 3(3°). Set

$$\alpha = (1+q)/(mq-1), \qquad \beta = (1+m)/(mq-1).$$

By the direct calculation, we see that (2.2) and Lemma 1(d) hold. Moreover, $p < 2q - \alpha/\beta$ since $p < (2mq+q-1)/(1+m)$. Therefore, (2.11) holds by Lemma 2. From the expressions of α and β we have

$$2(q\beta/\alpha - 1) = 2(mq-1)/(1+q), \qquad 2(q - \alpha/\beta) = 2(mq-1)/(1+m).$$

Consequently, (1.10) holds.

The proofs of cases (iii) and (iv) are analogous. □

3. Proof of Theorem 2

Since $g(t)$ is continuous, nondecreasing and $\lim_{t\to T^-} g(t) = \infty$. For any $t_0 \in (0, T)$, write

$$t_0^+ = t^+(t_0) = \max\bigl\{t \in (t_0, T) \mid g(t) = 2g(t_0)\bigr\}. \tag{3.1}$$

Choose $\lambda_0 = \lambda(t_0)$ as in (2.5).

The proof of Theorem 2 depends mainly on the two following lemmas.

Lemma 4. *Suppose that positive constants α and β satisfy the condition* (2.2) *of Lemma* 1, *and*

$$\max\{\alpha, \beta\} \geqslant N. \tag{3.2}$$

If one of the following holds:

(i) $2+\alpha-p\beta=2+\beta-k\alpha=0$,
(ii) $2+\alpha-p\beta=1+\beta-m\alpha=0$, *and when* $\max\{\alpha,\beta\}=N$ *we further assume that* $m, p \geqslant 1$,
(iii) $1+\alpha-q\beta=2+\beta-k\alpha=0$, *and when* $\max\{\alpha,\beta\}=N$ *we further assume that* $q, k \geqslant 1$,
(iv) $1+\alpha-q\beta=1+\beta-m\alpha=0$,

then there exists a positive constant M independent of t_0 such that

$$\lambda^{-2}(t_0)\big(t_0^+-t_0\big) \leqslant M, \quad \forall t_0 \in (T/2, T). \tag{3.3}$$

Moreover, there is a constant $C>0$ such that

$$f(t_0) \leqslant C(T-t_0)^{-\alpha/2}, \quad g(t_0) \leqslant C(T-t_0)^{-\beta/2}, \quad \forall t_0 \in (T/2, T). \tag{3.4}$$

Remark 3. If we take t_0^+ as in (3.1) and $\lambda(t_0)=f^{-1/\alpha}(t_0)$, then (3.3) and (3.4) are also true.

Proof of Lemma 4. If (3.3) were false, there would exist a sequence $\{t_n\}$ with $t_n \to T^-$ such that $\lambda_n^{-2}(t_n^+-t_n) \to \infty$, where $\lambda_n=\lambda(t_n)$ and $t_n^+=t^+(t_n)$. For each t_n, choose $(\hat{x}_n, \hat{t}_n) \in \partial\mathbb{R}_+^N \times (0, t_n]$ (the choice of $\hat{x}_n$ is possible due to (1.3) and the definition of g) such that $v(\hat{x}_n, \hat{t}_n) \geqslant g(t_n)/2$. Rescale (u, v) around $(\hat{x}_n, \hat{t}_n)$ as in (2.5), and then obtain a solution (φ_n, ψ_n) of (2.6) in $\mathbb{R}_+^N \times I_n(t_n^+)$. From (2.3) and the definition of t_n^+ we get

$$\begin{cases} \psi_n(0,0) \geqslant 1/2, \quad 0 \leqslant \psi_n(y,s) \leqslant 2, & y \in \overline{\mathbb{R}_+^N},\ s \in I_n(t_n^+), \\ 0 \leqslant \varphi_n(y,s) \leqslant \lambda_n^\alpha f(t_n^+) \leqslant \lambda_n^\alpha \varepsilon^{-\alpha} g^{\alpha/\beta}(t_n^+) = 2^{\alpha/\beta}\varepsilon^{-\alpha}, & y \in \overline{\mathbb{R}_+^N},\ s \in I_n(t_n^+). \end{cases}$$

Same as in the proof of Lemma 1, there exist $C^{2,1}$ functions $\varphi(y,s)$ and $\psi(y,s)$, which satisfy

$$\begin{cases} \varphi_s = \Delta_y\varphi + \delta_1\psi^p, \quad \psi_s = \Delta_y\psi + \delta_2\varphi^k, & y \in \mathbb{R}_+^N,\ s \in (-\infty, +\infty), \\ -\varphi_{y_1} = \delta_3\psi^q, \quad -\psi_{y_1} = \delta_4\varphi^m, & y_1 = 0,\ s \in (-\infty, +\infty), \end{cases} \tag{3.5}$$

and for $(y,s) \in \overline{\mathbb{R}_+^N} \times (-\infty, +\infty)$,

$$\psi(0,0) \geqslant 1/2, \quad 0 \leqslant \varphi \leqslant 2^{\alpha/\beta}\varepsilon^{-\alpha}, \quad 0 \leqslant \psi \leqslant 2, \tag{3.6}$$

where δ_i $(i=1,2,3,4)$ are nonnegative constants defined by (2.9). Notice that φ cannot be trivial, otherwise (3.5) would contradict to (3.6).

(i) If $2+\alpha-p\beta=2+\beta-k\alpha=0$, i.e., $\alpha=2(1+p)/(pk-1)$, $\beta=2(1+k)/(pk-1)$, then $\delta_1=\delta_2=1$, $\delta_3, \delta_4 \geqslant 0$. As $\max\{\alpha,\beta\} \geqslant N$, i.e., $\max\{1+p, 1+k\}/(pk-1) \geqslant N/2$, the results of [7] imply that (φ, ψ) of (3.5) blows up in finite time. It contradicts to (3.6). So, (3.3) holds.

(ii) If $2+\alpha-p\beta = 1+\beta-m\alpha = 0$, i.e., $\alpha = (p+2)/(mp-1)$, $\beta = (1+2m)/(mp-1)$, then $\delta_1 = \delta_4 = 1$, $\delta_2, \delta_3 \geqslant 0$. Using the results of [8] and (3.2), by the comparison principle, we know that (φ, ψ) of (3.5) blows up in finite time. A contradiction. Hence, (3.3) holds.

The proof of case (iii) is exactly the same as that of case (ii).

(iv) If $1+\alpha-q\beta = 1+\beta-m\alpha = 0$, i.e., $\alpha = (1+q)/(mq-1)$, $\beta = (1+m)/(mq-1)$, then $\delta_3 = \delta_4 = 1$, $\delta_1, \delta_2 \geqslant 0$. In view of (3.2) and the results of [5], the comparison principle asserts that (φ, ψ) of (3.5) blows up in finite time. It contradicts to (3.6). Therefore (3.3) holds.

From the above discussions we see that (3.3) holds for all $T/2 < t_0 < T$.

Next, we analysis (3.4). In view of (2.5) and (3.3) it follows that

$$t_0^+ - t_0 \leqslant M g^{-2/\beta}(t_0), \quad \forall t_0 \in (T/2, T).$$

Fix $t_0 \in (T/2, T)$ and put $t_1 = t_0^+$, $t_2 = t_1^+$, $t_3 = t_2^+, \ldots,$ where t_n^+ is defined as in (3.1), then

$$t_{j+1} - t_j \leqslant M g^{-2/\beta}(t_j), \quad g(t_{j+1}) = 2g(t_j), \quad j = 0, 1, 2, \ldots.$$

Consequently,

$$T - t_0 = \sum_{j=0}^{\infty}(t_{j+1} - t_j) \leqslant M \sum_{j=0}^{\infty} g^{-2/\beta}(t_j) = M g^{-2/\beta}(t_0) \sum_{j=0}^{\infty} 4^{-j/\beta},$$

which implies

$$g(t_0) \leqslant C(T - t_0)^{-\beta/2}, \quad t_0 \in (T/2, T) \tag{3.7}$$

with $C = (M \sum_{j=0}^{\infty} 4^{-j/\beta})^{\beta/2}$. This fact together with (2.3) yields

$$f(t_0) \leqslant \varepsilon^{-\alpha} g^{\alpha/\beta}(t_0) = \varepsilon^{-\alpha} C^{\alpha/\beta}(T - t_0)^{-\alpha/2}, \quad t_0 \in (T/2, T). \tag{3.8}$$

The estimates (3.7) and (3.8) assert (3.4). We complete Lemma 4. □

In the sequel, we show that Theorem 2 is true.

Proof of Theorem 2. (i) If $p \geqslant (2qk+2q-1)/(2+k)$ and $k \geqslant (2mp+2m-1)/(2+p)$, then $pk > 1$ by Lemma 3(1°). Put

$$\alpha = 2(1+p)/(pk-1), \qquad \beta = 2(1+k)/(pk-1).$$

Series of calculations tell us that (2.2), (3.2) and Lemma 4(i) hold. By Lemma 4 and (2.1), we reach (1.12).

Proofs of cases (ii)–(iv) are similar. Therefore, Theorem 2 is concluded. □

4. Proof of Theorem 3

To verify Theorem 3, we need the lemma below.

Lemma 5. *Let p, k, α and β be positive constants. Assume that positive functions u and v are continuous on $\overline{\mathbb{R}_+^N} \times [0, T)$ and satisfy*

$$\begin{cases} u_t = u_{xx} + v^p, \quad v_t = v_{xx} + u^k, & x \in \mathbb{R}_+^N,\ t \in (0, T), \\ u(x,t) \leqslant \frac{C}{(T-t)^\alpha}, \quad v(x,t) \leqslant \frac{C}{(T-t)^\beta}, & x \in \mathbb{R}_+^N,\ t \in (0, T), \end{cases} \tag{4.1}$$

for some positive constant C. If

$$p\beta - \alpha - 1 \leqslant 0, \qquad k\alpha - \beta - 1 \leqslant 0$$

and at least one of them is strict, then for any constant $a > 0$,

$$\sup\bigl\{u(x,t) + v(x,t):\ x_1 > a,\ 0 \leqslant t < T\bigr\} < \infty,$$

where $x = (x_1, x') \in \mathbb{R}_+^N$. This implies that blow-up may occur only on the boundary $\partial\mathbb{R}_+^N$.

Proof. The idea of this proof comes from [16]. Let A, B and M be positive constants and will be determined later. Set

$$\begin{cases} \varphi(x) = (1 - e^{-x_1})^2, & x \in \overline{\mathbb{R}_+^N}, \\ w(x,t) = \frac{AM^\alpha}{[\varphi(x) + M(T-t)]^\alpha}, & x \in \overline{\mathbb{R}_+^N},\ t \in [t_0, T), \\ z(x,t) = \frac{BM^\beta}{[\varphi(x) + M(T-t)]^\beta}, & x \in \overline{\mathbb{R}_+^N},\ t \in [t_0, T), \end{cases}$$

where t_0 $(0 < t_0 < T)$ satisfies $M(T - t_0) = 1$. From this it is obvious that t_0 varies with M. A direct calculation gives

$$\begin{cases} w(x,t_0) = \frac{AM^\alpha}{(\varphi(x)+1)^\alpha} \geqslant 2^{-\alpha} AM^\alpha = 2^{-\alpha} A(T - t_0)^{-\alpha}, & x \in \overline{\mathbb{R}_+^N}, \\ z(x,t_0) = \frac{BM^\beta}{(\varphi(x)+1)^\beta} \geqslant 2^{-\beta} BM^\beta = 2^{-\beta} B(T - t_0)^{-\beta}, & x \in \overline{\mathbb{R}_+^N}, \\ w(0,x',t) = \frac{A}{(T-t)^\alpha}, \quad z(0,x',t) = \frac{B}{(T-t)^\beta}, & x' \in \mathbb{R}^{N-1},\ t \in [t_0, T). \end{cases} \tag{4.2}$$

If we take A and B so large that

$$A \geqslant 2^\alpha C, \qquad B \geqslant 2^\beta C. \tag{4.3}$$

Then from (4.1) and (4.2) we have

$$\begin{cases} w(x,t_0) \geqslant u(x,t_0), \quad z(x,t_0) \geqslant v(x,t_0), & x \in \overline{\mathbb{R}_+^N}, \\ w(0,x';t) \geqslant u(0,x';t), \quad z(0,x';t) \geqslant v(0,x';t), & x' \in \mathbb{R}^{N-1},\ t \in [t_0, T). \end{cases} \tag{4.4}$$

By careful calculation, we have

$$\begin{cases} w_t - w_{xx} - z^p = \frac{\alpha A M^\alpha}{[\varphi(x)+M(T-t)]^{1+\alpha}}\left(M + \Delta\varphi - \frac{(1+\alpha)|\nabla\varphi|^2}{\varphi(x)+M(T-t)}\right) - z^p \\ \qquad \triangleq F(x,t)\frac{\alpha A M^\alpha}{[\varphi(x)+M(T-t)]^{1+\alpha}}, \\ z_t - z_{xx} - w^k = \frac{\beta B M^\beta}{[\varphi(x)+M(T-t)]^{1+\beta}}\left(M + \Delta\varphi - \frac{(1+\beta)|\nabla\varphi|^2}{\varphi(x)+M(T-t)}\right) - w^k \\ \qquad \triangleq G(x,t)\frac{\beta B M^\beta}{[\varphi(x)+M(T-t)]^{1+\beta}}, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} F(x,t) &= M + \Delta\varphi - \frac{(1+\alpha)|\nabla\varphi|^2}{\varphi(x)+M(T-t)} - \frac{B^p M^{p\beta-\alpha}}{\alpha A[\varphi(x)+M(T-t)]^{p\beta-\alpha-1}} \\ &= M + 2e^{-x_1}(2e^{-x_1}-1) - \frac{4(1+\alpha)e^{-2x_1}\varphi(x)}{\varphi(x)+M(T-t)} \\ &\quad - \frac{B^p M^{p\beta-\alpha}}{\alpha A[\varphi(x)+M(T-t)]^{p\beta-\alpha-1}} \\ &\geqslant M - 2(3+2\alpha) - \frac{B^p M^{p\beta-\alpha}}{\alpha A[\varphi(x)+M(T-t)]^{p\beta-\alpha-1}}, \\ G(x,t) &= M + \Delta\varphi - \frac{(1+\beta)|\nabla\varphi|^2}{\varphi(x)+M(T-t)} - \frac{A^k M^{k\alpha-\beta}}{\beta B[\varphi(x)+M(T-t)]^{k\alpha-\beta-1}} \\ &\geqslant M - 2(3+2\beta) - \frac{A^k M^{k\alpha-\beta}}{\beta B[\varphi(x)+M(T-t)]^{k\alpha-\beta-1}}. \end{aligned}$$

Recalling that $\varphi(x) \leqslant 1$, $M(T-t) \leqslant 1$ for $x \in \overline{\mathbb{R}_+^N}$ and $t \in [t_0, T)$. If we choose $M \geqslant 4(3+2\alpha+2\beta)$, then, by using of $1+\alpha-p\beta \geqslant 0$ and $1+\beta-k\alpha \geqslant 0$, we have

$$\begin{aligned} F(x,t) &\geqslant M/2 - \alpha^{-1}A^{-1}B^p M^{p\beta-\alpha}\big[\varphi(x)+M(T-t)\big]^{1+\alpha-p\beta} \\ &\geqslant M/2 - \alpha^{-1}A^{-1}B^p M^{p\beta-\alpha}2^{1+\alpha-p\beta} \\ &= M(1/2 - \alpha^{-1}A^{-1}B^p M^{p\beta-\alpha-1}2^{1+\alpha-p\beta}), \quad x \in \overline{\mathbb{R}_+^N},\ t \in [t_0, T), \end{aligned} \tag{4.6}$$

$$G(x,t) \geqslant M(1/2 - \beta^{-1}B^{-1}A^k M^{k\alpha-\beta-1}2^{1+\beta-k\alpha}), \quad x \in \overline{\mathbb{R}_+^N},\ t \in [t_0, T). \tag{4.7}$$

If $p\beta-\alpha-1<0$ and $k\alpha-\beta-1<0$, we may first fix A and B such that (4.3) holds (and hence (4.4) holds), and then choose M large enough such that the right-hand sides of (4.6) and (4.7) are positive. For such A, B, M and $t_0 = T-1/M$, we have, by (4.5),

$$w_t \geqslant w_{xx} + z^p, \quad z_t \geqslant z_{xx} + w^k, \quad x \in \overline{\mathbb{R}_+^N},\ t \in [t_0, T). \tag{4.8}$$

If $p\beta-\alpha-1=0$ and $k\alpha-\beta-1<0$, for the fixed $B \geqslant 2^\beta C$, we first take A ($A \geqslant 2^\alpha C$) so large that the right-hand side of (4.6) is positive, and then choose M large enough such that the right-hand side of (4.7) is positive. Therefore, (4.4) and (4.8) hold for such chosen A, B, M and $t_0 = T-1/M$.

If $p\beta-\alpha-1<0$ and $k\alpha-\beta-1=0$, by the same way as in the above arguments, we see that there exist positive constants A, B and M and $t_0 = T-1/M$ such that (4.4) and (4.8) hold.

Applying the *comparison principle*, it follows from (4.1), (4.4) and (4.8) that

$$u(x,t) \leqslant w(x,t) = \frac{AM^{\alpha}}{[\varphi(x) + M(T-t)]^{\alpha}}, \quad x \in \overline{\mathbb{R}^N_+},\ t \in [t_0, T),$$
$$v(x,t) \leqslant z(x,t) = \frac{BM^{\beta}}{[\varphi(x) + M(T-t)]^{\beta}}, \quad x \in \overline{\mathbb{R}^N_+},\ t \in [t_0, T).$$

The conclusion of this lemma holds. □

Proof of Theorem 3. (a) When $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k < (2mp + 2m - 1)/(2 + p)$. For the case $p \geqslant (2mq + q - 1)/(1 + m)$, if $\max\{p + 2, 1 + 2m\}/(mp - 1) > N$, or $\max\{p + 2, 1 + 2m\}/(mp - 1) = N$ and $m, p \geqslant 1$, let $\alpha = (2 + p)/[2(mp - 1)]$ and $\beta = (1+2m)/[2(mp-1)]$. Then we have (1.13), which implies (4.1). A direct computation gives

$$p\beta - \alpha - 1 = 0, \qquad k\alpha - \beta - 1 = \big[k(2 + p) - (2mp + 2m - 1)\big]/2(mp - 1) < 0$$

since $k(2 + p) < 2mp + 2m - 1$. Lemma 5 shows that the conclusion of Theorem 3 holds.

For the case $p < (2mq + q - 1)/(1 + m)$ and $\max\{1 + q, 1 + m\}/(mq - 1) \geqslant N$. By Lemma 3(3°), $k < (2mq + m - 1)/(1 + q)$. Let $\alpha = (1 + q)/[2(mq - 1)]$ and $\beta = (1+m)/[2(mq-1)]$. Then (1.14), and thus, (4.1) holds. Applying $p(1+m) < 2mq + q - 1$ and $k(1 + q) < 2mq + m - 1$, direct computations show that

$$p\beta - \alpha - 1 = \frac{p(1 + m) - (2mq + q - 1)}{2(mq - 1)} < 0,$$
$$k\alpha - \beta - 1 = \frac{k(1 + q) - (2mq + m - 1)}{2(mq - 1)} < 0.$$

The conclusion follows from Lemma 5.

(b) When $p < (2qk + 2q - 1)/(2 + k)$ and $k \geqslant (2mp + 2m - 1)/(2 + p)$. For the case $k \geqslant (2mq + m - 1)/(1 + q)$, if $\max\{1 + 2q, 2 + k\}/(qk - 1) > N$, or $\max\{1 + 2q, 2 + k\}/(qk - 1) = N$ and $q, k \geqslant 1$, let $\alpha = (1 + 2q)/[2(qk - 1)]$ and $\beta = (2 + k)/[2(qk - 1)]$. Then (1.15), and hence, (4.1) holds. Adopting similar arguments as above we know that Theorem 3 holds.

For the other cases, the proofs are similar. □

5. Further discussion for one-dimensional case

To get the upper bounds of blow-up rates in Theorem 2 and thus blow-up sets in Theorem 3, the parameters p, q, k and m should satisfy some additional conditions that depend on the dimension N, even in the case $N = 1$. For example, in Theorem 2(ii), if $(2qk + 2q - 1)/(2 + k) \leqslant p < (2mq + q - 1)/(1 + m)$, $k < (2mp + 2m - 1)/(2 + p)$, and $N = 1$, to guarantee (1.14) holds we need also to assume that $\max\{1 + q, 1 + m\}/(mq - 1) \geqslant 1$. In this part, we discuss further the upper bounds of blow-up rates and the blow-up set for the case $N = 1$, and give more perfect conclusions.

Theorem 4. *Suppose that* $u_0, v_0 \in C^2(0,\infty) \cap L^{\infty}(0,\infty)$ *and satisfy*

$$\begin{cases} u_0(x), v_0(x) \geqslant 0, & \lim_{x\to\infty} u_0(x) = \lim_{x\to\infty} v_0(x) = 0, & & x > 0, \\ u_0'(x), v_0'(x) \leqslant 0, & u_0''(x) + v_0^p(x) \geqslant 0, & v''(x) + u_0^k(x) \geqslant 0, & x > 0. \end{cases} \tag{5.1}$$

Then there exist positive constants c and C such that the following hold.

(i) *When $p \geqslant (2qk+2q-1)/(2+k)$ and $k \geqslant (2mp+2m-1)/(2+p)$, we have*

$$\begin{cases} c \leqslant u(0,t)(T-t)^{(1+p)/(pk-1)} \leqslant C, & 0<t<T, \\ c \leqslant v(0,t)(T-t)^{(1+k)/(pk-1)} \leqslant C, & 0<t<T. \end{cases} \tag{5.2}$$

(ii) *When $p \geqslant (2qk+2q-1)/(2+k)$ and $k < (2mp+2m-1)/(2+p)$. If $p \geqslant (2mq+q-1)/(1+m)$, then*

$$\begin{cases} c \leqslant u(0,t)(T-t)^{(2+p)/[2(mp-1)]} \leqslant C, & 0<t<T, \\ c \leqslant v(0,t)(T-t)^{(1+2m)/[2(mp-1)]} \leqslant C, & 0<t<T; \end{cases} \tag{5.3}$$

if $p < (2mq+q-1)/(1+m)$, then

$$\begin{cases} c \leqslant u(0,t)(T-t)^{(1+q)/[2(mq-1)]} \leqslant C, & 0<t<T, \\ c \leqslant v(0,t)(T-t)^{(1+m)/[2(mq-1)]} \leqslant C, & 0<t<T. \end{cases} \tag{5.4}$$

(iii) *When $p < (2qk+2q-1)/(2+k)$ and $k \geqslant (2mp+2m-1)/(2+p)$. If $k \geqslant (2mq+m-1)/(1+q)$, then*

$$\begin{cases} c \leqslant u(0,t)(T-t)^{(1+2q)/[2(qk-1)]} \leqslant C, & 0<t<T, \\ c \leqslant v(0,t)(T-t)^{(2+k)/[2(qk-1)]} \leqslant C, & 0<t<T; \end{cases} \tag{5.5}$$

if $k < (2mq+m-1)/(1+q)$, then (5.4) *holds.*

(iv) *When $p < (2qk+2q-1)/(2+k)$ and $k < (2mp+2m-1)/(2+p)$,* (5.4) *holds.*

Proof. By the maximum principle and (5.1), we infer that

$$u_t, v_t \geqslant 0, \quad u_x, v_x \leqslant 0, \quad x \geqslant 0, \ t \in [0,T).$$

Hence, $u(0,t) = f(t) = \max_{[0,t]\times[0,\infty)} u(x,\tau)$, $v(0,t) = g(t) = \max_{[0,t]\times[0,\infty)} v(x,\tau)$ for $t \in [0,T)$. The lower bound estimates come directly from Theorem 1, while the upper bound estimates derive from Lemma 3 and the following lemma. $\square$

Lemma 6. *Let positive constants α and β satisfy the conditions of Lemma* 1. *Under the assumptions of Theorem* 4, *we have the following conclusions.*

(A) *If $p \leqslant 2q - \alpha/\beta$, then*

$$\begin{aligned} &f(z) \leqslant C(T-z)^{-1/[2(q\beta/\alpha-1)]}, \\ &g(z) \leqslant C(T-z)^{-1/[2(q-\alpha/\beta)]}, \quad T/2<z<T, \end{aligned} \tag{5.6}$$

provided that $q\beta/\alpha > 1$.

(B) *If $k \leqslant 2m - \beta/\alpha$, then*

$$\begin{aligned} &g(z) \leqslant C(T-z)^{-1/[2(m\alpha/\beta-1)]}, \\ &f(z) \leqslant C(T-z)^{-1/[2(m-\beta/\alpha)]}, \quad T/2<z<T, \end{aligned} \tag{5.7}$$

provided that $m\alpha/\beta > 1$.

(C) *If* $p \geqslant 2q - \alpha/\beta$ *and* $k \geqslant 2m - \beta/\alpha$, *then*

$$\begin{cases} g(z) \leqslant C(T-z)^{-1/[(1+2k)\alpha/\beta - p - 2]}, & T/2 < z < T, \\ f(z) \leqslant C(T-z)^{-1/[1+2k-(2+p)\beta/\alpha]}, & T/2 < z < T, \end{cases} \tag{5.8}$$

provided that $p\beta/\alpha > 1$ *and* $(k+1/2)\alpha/\beta - p/2 > 1$.

Proof. Remember that this time the Green's function $G(x, y; t)$ for the heat equation in R_+ satisfying $\partial G/\partial y = 0$ at $y = 0$ is written as

$$G(x, y; t) = (4\pi t)^{-1/2}\left\{\exp\left(-\frac{(x-y)^2}{4t}\right) + \exp\left(-\frac{(x+y)^2}{4t}\right)\right\}.$$

The representation formulae (see [5]) for the solution of (1.1) is

$$\begin{aligned} u(x,t) &= \int_0^\infty G(x,y;t)u_0(y)\,dy + \int_0^t \int_0^\infty G(x,y;t-\tau)v^p(y,\tau)\,dy\,d\tau \\ &\quad + \int_0^t G(x,0;t-\tau)v^q(0,\tau)\,d\tau, \\ v(x,t) &= \int_0^\infty G(x,y;t)v_0(y)\,dy + \int_0^t \int_0^\infty G(x,y;t-\tau)u^k(y,\tau)\,dy\,d\tau \\ &\quad + \int_0^t G(x,0;t-\tau)u^m(0,\tau)\,d\tau. \end{aligned} \tag{5.9}$$

(A) When $p \leqslant 2q - \alpha/\beta$ and $q\beta/\alpha > 1$. It follows from (5.9) and (2.3) that

$$u(0,t) \geqslant \int_0^t G(0,0;t-\tau)v^q(0,\tau)\,d\tau \geqslant \int_0^t \varepsilon^{q\beta}\big(\pi(t-\tau)\big)^{-1/2}u^{q\beta/\alpha}(0,\tau)\,d\tau.$$

Proceeding as the proof of Lemma 2 in paper [27], we see that there exists a constant $C > 0$ such that

$$f(t) = u(0,t) \leqslant C(T-t)^{-1/[2(q\beta/\alpha - 1)]}.$$

It means that the first inequality of (5.6) is true. Combing this fact with (2.3) yields the second one of (5.6).

(B) When $k \leqslant 2q - \beta/\alpha$ and $m\alpha/\beta > 1$. The proof is analogous as that of case (A).

(C) When $p \geqslant 2q - \alpha/\beta$, $k \geqslant 2m - \beta/\alpha$, $p\beta/\alpha > 1$ and $(k+1/2)\alpha/\beta - p/2 > 1$. We adopt a similar analysis as that of [24]. Since $u_{xx} + v^p = u_t \geqslant 0$,

$$u_{xx} \geqslant -v^p(x,t) \geqslant -v^p(0,t), \quad x \geqslant 0,\ T/2 \leqslant t < T.$$

Recalling (2.3), it follows that for some constants $c, C > 1$ (we may assume that $u(0,t) > 1$ for all $T/2 \leqslant t < T$),

$$
\begin{aligned}
u_x(x,t) &= u_x(0,t) + \int_0^x u_{xx}\,dx \geqslant -v^q(0,t) - xv^p(0,t) \\
&\geqslant -cu^{q\beta/\alpha}(0,t) - Cxu^{p\beta/\alpha}(0,t), \quad x \geqslant 0,\ T/2 \leqslant t < T.
\end{aligned}
$$

Integrating the above inequality from 0 to $x > 0$ we have

$$
u(x,t) \geqslant u(0,t) - cxu^{(1+p\beta/\alpha)/2}(0,t) - \frac{1}{2}Cx^2u^{p\beta/\alpha}(0,t), \quad x \geqslant 0,\ T/2 \leqslant t < T.
$$

Therefore,

$$
u(x,t) \geqslant u(0,t)/2 \quad \text{for } 0 \leqslant x \leqslant \mu(t),\ T/2 \leqslant t < T, \tag{5.10}
$$

provided that

$$
c\mu(t)u^{(1+p\beta/\alpha)/2}(0,t) \leqslant \frac{1}{4}u(0,t), \qquad \frac{1}{2}C\mu^2(t)u^{p\beta/\alpha}(0,t) \leqslant \frac{1}{4}u(0,t).
$$

One can simply take $\mu(t) = (1/(4C_3))u^{(1-p\beta/\alpha)/2}(0,t)$ with $C_3 = \max\{c, C\}$. Then (5.10) holds. Note that $f(t) = u(0,t)$ and $g(t) = v(0,t)$, in view of (5.10), it follows from (5.9) that

$$
\begin{aligned}
v(0,t) &\geqslant \int_z^t \int_0^\infty G(0,y;t-\tau)u^k(y,\tau)\,dy\,d\tau \\
&\geqslant \int_z^t \big\{u(0,\tau)/2\big\}^k \left(\int_0^{\mu(\tau)} G(0,y;t-\tau)\,dy \right) d\tau.
\end{aligned} \tag{5.11}
$$

A direct calculation illustrates

$$
\int_0^{\mu(\tau)} G(0,y;t-\tau)\,dy = \int_0^{\mu(\tau)/2\sqrt{t-\tau}} \frac{2e^{-y^2}}{\sqrt{\pi}}\,dy \geqslant \int_0^{\mu(\tau)/2\sqrt{T-\tau}} \frac{2e^{-y^2}}{\sqrt{\pi}}\,dy. \tag{5.12}
$$

The first inequality of (2.10) asserts that

$$
\frac{\mu(\tau)}{2\sqrt{T-\tau}} = \frac{1}{8C_3\sqrt{T-\tau}}u^{(1-p\beta/\alpha)/2}(0,\tau) \leqslant \frac{1}{8C_3}c^{(1-p\beta/\alpha)/2} \triangleq C_4,
$$

and thus, by (5.12),

$$
\begin{aligned}
\int_0^{\mu(\tau)} G(0,y;t-\tau)\,dy &\geqslant \int_0^{\mu(\tau)/2\sqrt{T-\tau}} \frac{2e^{-C_4^2}}{\sqrt{\pi}}\,dy \\
&= \frac{c_2}{\sqrt{T-\tau}}u^{(1-p\beta/\alpha)/2}(0,\tau),
\end{aligned} \tag{5.13}
$$

where c_2 is a positive constant. Combining (5.11) with (5.13) and using (2.3) we have

$$v(0,t) \geqslant 2^{-k} c_2 \int_z^t \frac{1}{\sqrt{T-\tau}} u^{k+(1-p\beta/\alpha)/2}(0,\tau)\, d\tau$$
$$\geqslant c_2 c_3 \int_z^t \frac{1}{\sqrt{T-\tau}} v^{[k+(1-p\beta/\alpha)/2]\alpha/\beta}(0,\tau)\, d\tau$$

for some positive constant c_3. Same as the proof of Lemma 2 of [27], we know that there exists positive constant C such that

$$g(t) = v(0,t) \leqslant C(T-t)^{-1/\{2[(k+1/2)\alpha/\beta - p/2 - 1]\}} = C(T-t)^{-1/[(1+2k)\alpha/\beta - p - 2]}.$$

This is just the first inequality of (5.8). Applying the second inequality of (2.3) we obtain the second inequality of (5.8). The proof of Lemma 6 is completed. □

Using Lemma 5 and Theorem 4, analogous as the proof of Theorem 3, for $N = 1$ we obtain a version of Theorem 3.

Theorem 5. *If the inequalities $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k \geqslant (2mp + 2m - 1)/(2 + p)$ do not hold simultaneously, then the blow-up set consists of a single point $x = 0$.*

Remark 4. If $p \geqslant (2qk + 2q - 1)/(2 + k)$ and $k \geqslant (2mp + 2m - 1)/(2 + p)$, we do not know whether or not Theorem 5 holds. It is still an open problem.

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